# DYNAMIC THEORY OF A CONTROLLED VIBRATION DAMPING MODEL

(DINAMICHESKAIA TEORIIA ODNOI MODELI UPRAVLIAEMOGO VIBROGASHENIIA)

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The need for efficient damping of vibrations over a broad range of frequencies and amplitudes dictates the use of controlled vibration dampers in many situations. Several such devices are described in [1 to 4]. The present paper concerns the dynamics of an impact-type controlled vibration damping model.

Applying the usual simplifications, we shall consider the problem of forced vibrations of a nonlinear system with controlled interactions of the absolutely inelastic collision type occurring between its two masses. The instants at which these interactions take place are determined by the position of the principal mass of the system only. Computations carried out by the method of point transformations with the aid of bifurcation theory are used to determine the amplitude characteristics of the double-impact symmetrical periodic motion. The domain of existence and stability of this motion turns out to be considerably larger than in the case of an ordinary impact-type vibration damper [5 to 7] in which collision interactions occur in configurations determined by the relative positions of both masses. With a relative damper mass $\mu < 4/(\pi^2 - 4) \approx 0.68$, the double-impact mode is stable over the entire frequency range. This fact is extremely significant, since in other modes the interactions can be the cause of dangerous resonance vibrations [7 and 8].

The vibration damping model considered is much more efficient than that of an optimum tuned Lanchester viscous damper [9], as well as of a damper with an extremal damping factor control [3].

**1. The equations of motion.** The vibrational system which we shall consider consists of an elastically fastened principal mass $M$ subjected to a force of the form $F\cos\Omega t$. Atop the mass $M$ is a second mass $m$ which can move freely relative to the principal mass in the direction of action of the external force. When the mass $M$ passes through the position $\xi = 0$ corresponding to the undeformed state of the elastic fastening, the masses experience momentary coupling, i.e. an interaction of the absolutely inelastic collision type (one way of accomplishing this coupling is by transmission of a pulse into the winding of an electromagnet). Let us write out the equations of motion of the model neglecting frictional forces

$$M\xi^{\cdot\cdot} + k\xi = F\cos\Omega t, \quad \eta^{\cdot\cdot} = 0 \qquad (\xi \neq 0) \tag{1.1}$$

$$\xi_+^{\cdot} = \eta_+^{\cdot} = \frac{M\xi_-^{\cdot} + m\eta_-^{\cdot}}{M + m} \qquad (\xi = 0) \tag{1.2}$$

In these equations the displacement of the mass $M$ is denoted by $\xi$. The velocities of the masses immediately before and immediately after their interaction are given by $\xi_-^\cdot$, $\eta_-^\cdot$ and $\xi_+^\cdot$, $\eta_+^\cdot$, respectively. The coefficient $k$ characterizes the rigidity of the elastic coupling.

By introducing the dimensionless variables and time

$$x = \xi k / F, \quad y = \eta k / F, \quad \tau = t\sqrt{k/M} \tag{1.3}$$

we can write the initial equations in simpler form,

$$x^{\cdot\cdot} + x = \cos\omega\tau, \qquad y^{\cdot\cdot} = 0. \qquad (x \neq 0) \tag{1.4}$$

$$x_+^\cdot = y_+^\cdot = \frac{x_-^\cdot + \mu y_-}{1+\mu} \qquad (x = 0) \tag{1.5}$$

The dimensionless parameters $\mu$ and $\omega$ are related to the dimensional parameters by the relations

$$= m / M, \qquad \omega = \Omega\sqrt{M/k}$$

**2. The domain of existence and stability of the periodic motion.** The motions described by Equations (1.4) occur in the planes $y^\cdot$ = const of the four-dimensional phase space $x$, $x^\cdot$, $y^\cdot$, $\tau$. At $x = 0$ there occurs instantaneous transition of the phase point from one plane $y^\cdot$ = const to another. In order to study the periodic motions, let us consider a point transformation into itself of the surface $x = 0$ of interaction of masses as defined by Equations (1.4) and (1.5).

Let the initial point $M_0\{x_0 = 0,\ x_0^\cdot = y_0^\cdot,\ \tau_0\}$ of the transformation correspond to the instant directly following the interaction of the masses, and let the transformed point $M_1\ \{x_1 = 0,\ x_1^\cdot = y_1^\cdot,\ \tau_1\}$ correspond to the instant immediately following their next interaction. Since the interactions occur in accordance with condition (1.5) and since the dynamic model behaves like a single mass immediately after the collision, the problem reduces to the investigation of stationary transformation points of the two-dimensional surfaces $x^\cdot$, $\tau$. The relationship between the coordinates of the initial and final points of the transformation can be obtained directly by solving Equations (1.4) with allowance for (1.5). It turns out to be

$$(x_0^\cdot + \omega b\sin\omega\tau_0)\sin(\tau_1 - \tau_0) - b\cos\omega\tau_0\cos(\tau_1 - \tau_0) + b\cos\omega\tau_1 = 0$$

$$(1+\mu)\,x_1^\cdot = \mu x_0^\cdot + (x_0^\cdot + \omega b\sin\omega\tau_0)\cos(\tau_1 - \tau_0) + \tag{2.1}$$

$$+ b\cos\omega\tau_0\sin(\tau_1 - \tau_0) - \omega b\sin\omega\tau_1 \qquad (b = 1/(1-\omega^2))$$

The first equation of (2.1) specifies some instant $\tau_1$ of interaction between the masses satisfying the condition $\tau_1 > \tau_0$. The second equation can be used to find the transformed coordinate $x_1^\cdot$.

The coordinates of the stationary transformation point corresponding to the symmetrical double-impact periodic motion can be found from Equations (2.1) together with the conditions

$$x_1^\cdot = -x_0, \qquad \tau_1 = \tau_0 + \frac{\pi}{\omega} \tag{2.2}$$

After some simple transformations the required relations become

$$\lambda\omega\tan\omega\tau_0\tan\frac{\pi}{2\omega} = 1, \qquad x_0^\cdot = \omega b(\lambda - 1)\sin\omega\tau_0 \qquad \left(\lambda = \frac{\mu}{1+\mu}\right) \tag{2.3}$$

In order to isolate the domain of existence and stability of the periodic mode under investigation in the parameter plane $\lambda$, $\omega$, let us investigate the stability of the stationary transformation points found above. Varying point transformation equations (2.1) in the neighborhood of these points and setting

$$\delta\tau_1 = z\delta\tau_0, \qquad \delta x_1^{\cdot} = -z\delta x_0^{\cdot}$$

we obtain the characteristic equation

$$(1+\lambda)z^2 + \left[(2-\lambda)\cos\frac{\pi}{\omega} + \lambda + (\lambda\omega)^2 \tan\frac{\pi}{2\omega}\sin\frac{\pi}{\omega}\right]z +$$
$$+ (\lambda\omega)^2 \tan\frac{\pi}{2\omega}\sin\frac{\pi}{\omega} + \lambda\cos\frac{\pi}{\omega} - 2\lambda + 1 = 0 \qquad (2.4)$$

The periodic motion is stable if the roots of Equation (2.4) lie within a unit circle for the parameters $\omega$ and $\lambda$ corresponding to this motion. For $z = +1$ instability can arise on the boundary $N_+$ whose equation can be obtained from (2.4) by substituting in $z = +1$

$$1 + \lambda^2\omega^2 \tan^2\frac{\pi}{2\omega} = 0 \qquad (2.5)$$

For $z = -1$ and $z = e^{j\varphi}$ instability arises on the boundaries $N_-$ and $N_\varphi$, respectively. The equations of these boundaries are

$$\lambda = 1, \qquad \omega = 1/2,\ 1/4,\ 1/6,\ \ldots \qquad (2.6)$$

$$\lambda = 0, \qquad \lambda\omega^2\sin^2\frac{\pi}{\omega} = \left(3 - \cos\frac{\pi}{\omega}\right)\left(1 + \cos\frac{\pi}{\omega}\right) \qquad (2.7)$$

Since Equation (2.5) cannot be satisfied by real values of $\lambda$ and $\omega$, the domain of stability is bounded only by the segments $N_-$ and $N_\varphi$. Verification of the stability of the periodic motion for specific parameter values showed that the domain of stability is defined by the inequalities

$$0 < \lambda < \min\left\{1,\ \frac{(3-\cos\pi/\omega)(1+\cos\pi/\omega)}{\omega^2\sin^2\pi/\omega}\right\} \qquad (\omega > 0) \qquad (2.8)$$

The indicated domain of parameter values is shown in Fig. 1. We note that for $\lambda < 4/\pi^2$, i.e. for

$$\frac{m}{M} < \frac{4}{\pi^2 - 4} \approx 0.68$$

the periodic motion under investigation is stable over the entire frequency range.

**3. Violation of the conditions of existence due to additional interactions. Special bifurcation points.** Additional interactions of the masses can result not only in loss of stability, but also in disruption of the double-impact periodic motion. This occurs when the function $x(\tau_0 + \tau')$, given in accordance with (2.1) by Equation

$$x(\tau_0 + \tau') = (x_0^{\cdot} + \omega b \sin\omega\tau_0)\ \sin\tau' - b\cos\omega\tau_0\cos\tau' + b\cos\omega(\tau_0 + \tau') \qquad (3.1)$$

vanishes within the time interval $[\tau_0, \tau_0 + \pi/\omega]$.

One of the causes of additional interactions can be a change in the sign of the post-impact velocity

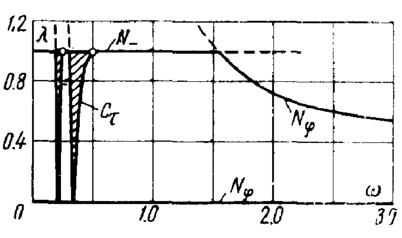


Fig. 1

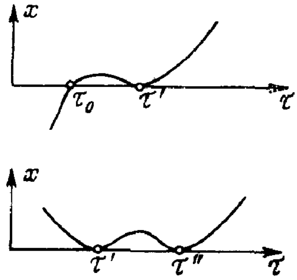


Fig. 2

$x_0^{\cdot}$ given by Equation (2, 3). But the relationship between the pre- and post-impact velocities for the periodic motion under investigation, i. e.

$$x_-^{\cdot} = (1 + 2\mu)\, x_0^{\cdot} \qquad (3.2)$$

implies that these velocities are always of the same sign.

Another reason for the appearance of additional interactions is the attainment of zero value by the extremum point of the function $x(\tau')$. The equation of the corresponding existence boundary $C_\tau$ in the parameter plane can be derived from the conditions that

$$x(\tau_0 + \tau') = 0, \quad x^{\cdot}(\tau_0 + \tau') = 0 \qquad (3.3)$$

at the instant $\tau_0 + \tau'$.

Making use of Equations (2, 1) and (2, 3), we can write conditions (3, 3) in the form

$$-\omega \tan \frac{\pi}{2\omega} + \sin \omega\tau' \cos \tau' - \omega \cos \omega\tau' \sin \tau' + (\sin \omega\tau' \sin \tau' + \omega \cos \omega\tau' \cos \tau') \tan \frac{\pi}{2\omega} = 0$$

$$\lambda \left[\cos \tau' + (\sin \tau' - \omega \sin \omega\tau') \tan \frac{\pi}{2\omega}\right] - \cos \omega\tau' = 0 \qquad \left(0 \leqslant \tau' \leqslant \frac{\pi}{\omega}\right) \qquad (3.4)$$

The investigation of transcendental equations (3, 4) can be simplified substantially by finding the special bifurcation points at which the bifurcation curves $C_\tau$ originate. To this end Equations (3, 4) must be supplemented by a condition of "creation" or "disappearance" of curves $C_\tau$. Fig. 2 shows two possible cases of behavior of the function $x(\tau')$ prior to the disappearance of $C_\tau$.

The first case corresponds to the merging of the tangent point $x(\tau')$ with the stationary transformation point : $\tau' \to \tau_0$. Analytically, this condition can be written as

$$x(\tau_0) = 0, \quad x^{\cdot}(\tau_0) = 0, \quad x^{\cdot\cdot}(\tau_0) = 0 \qquad (3.5)$$

The second case corresponds to the merging of the two tangent points : $\tau' \to \tau''$. Here the function and its first three derivatives vanish at the instant $\tau'$,

$$x(\tau') = 0, \qquad x^{\cdot}(\tau') = 0, \qquad x^{\cdot\cdot}(\tau') = 0, \qquad x^{\cdot\cdot\cdot}(\tau') = 0 \qquad (3.6)$$

After some simple transformations taking account of (2, 3) and (3 2), conditions (3, 5) and (3, 6) imposed on function (3, 1) yield, in the first case, the following values for the coordinates of the special bifurcation points:

$$\omega = {}^1/_2, {}^1/_4, {}^1/_6, \ldots, \lambda = 1, \qquad \omega = {}^1/_2, {}^1/_4, {}^1/_6, \ldots, \lambda = -1 \qquad (3.7)$$

In the second case they yield the incompatible equations

$$\cos \omega (\tau_0 + \tau') = 0, \qquad \sin \omega (\tau_0 + \tau') = 0$$

Hence, the boundaries $C_\tau$ of the domain of existence of the mode under investigation vanish when the tangency point merges with the stationary transformation point. The existence of bifurcation curves $C_\tau$ closed, or originating and ending at infinity, remains a possibility. Computations carried out on a "Razdan-2" computer indicated the absence of such boundaries. As regards the curves $C_\tau$ originating at points (3, 7), these isolate narrow domains of existence of more complex modes in the neighborhood of the frequencies $\omega = {}^1/_3, {}^1/_5, {}^1/_7, \ldots$ (Fig. 1).

**4. Amplitude of the forced vibrations. Comparative estimate of the vibration damping efficiency.** The displacement $x(\tau)$ of the principal mass of the vibrating system during the half-period $0 \leq \tau - \tau_0 \leq \pi/\omega$ is determined by Expression (3, 1) and by the values of $\tau_0$ and $x_0^{\cdot}$ as found from Equations (2, 3). The displacement $x(\tau)$ is equal to zero at the beginning and end of the indicated half-period. The amplitude value $X$ is attained at some instant $T^*$. The dependence $X(\omega)$ of the forced vibration amplitude on frequency computed electronically for various values of $\mu$ appear in Fig. 3 .

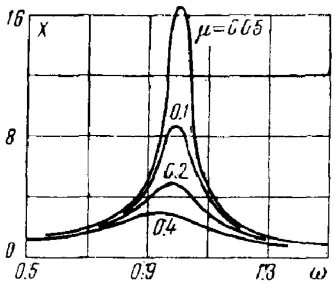

Fig. 3

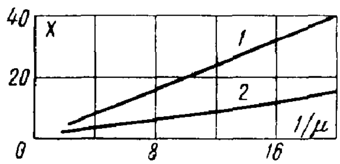

Fig. 4

As we know, the application of an ordinary impact damper can markedly reduce vibrations for specific frequencies and amplitudes of the disturbing force ; conversely, it can sharply increase vibrations for other frequencies and amplitudes of this force [7 and 8]. The results obtained indicate the possibility of efficient impact vibration damping over a broad range of perturbations. This is due to the fact that impact interaction occurs at instants determined by positions of the principal mass at which its velocity is close to maximum.

It is interesting to compare the efficiency of vibration dampers with different types of coupling between the mass of the damped system and the damper mass. Fig. 4 shows the vibration amplitude at resonance frequency as a function of the relative damper mass for an optimally tuned viscous-damped Lanchester damper [9] or a damper with an extremal damping factor control [3] (curve 1), and for the model investigated above (curve 2). As is evident from the figures, damping by means of controlled impact interaction is more than twice as efficient.

It should be noted that the strongly nonlinear system with impact interactions which we have considered cannot be "linearized" by introducing some equivalent viscous friction coefficient, since the divergence of the results is too large even with optimun linear friction.

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# CONVERGING CYLINDRICAL AND SPHERICAL DETONATION WAVES

(SKHODIASHCHIESIA TSILINDRICHESKIE I SPHERICHESKIE DETONATSIONNYE VOLNY)

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1. Spherical and cylindrical detonation waves, converging respectively to a point or to the axis of symmetry, are investigated. The usual assumptions are made :

1) the detonation wave is strong, i. e. the values of the pressure and internal energy in the undisturbed fluid can be neglected in comparison with their values in the disturbed gas ;

2) during the passage of the shock wave through the medium there is instantaneously released an energy $Q$, in $m^2/\sec^2$ (the magnitude of $Q$ refers to unit mass) ;

3) the process in the disturbed fluid is polytropic with exponent $\gamma$.

From the conditions for the conservation of mass, momentum, and energy at the detonation wave [1 and 2] we have

$$v_2 = \beta D, \qquad p_2 = \beta \rho_1 D^2, \qquad \rho_2 = \rho_1/(1-\beta) \tag{1.1}$$

$$\beta = \frac{1}{\gamma+1}\left\{1+\left[1-2(\gamma-1)(\gamma+1)\frac{Q}{D^2}\right]^{1/2}\right\} = \frac{\alpha}{\gamma+1} \tag{1.2}$$

The equations for the one-dimensional motion of a gas have the form, in Eulerian variables,

$$\rho\frac{\partial v}{\partial t} + \rho v\frac{\partial v}{\partial r} + \frac{\partial p}{\partial r} = 0, \qquad \frac{\partial \rho}{\partial t} + \rho\frac{\partial v}{\partial r} + v\frac{\partial \rho}{\partial r} + \frac{\nu\rho v}{r} = 0 \tag{1.3}$$

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial r} + \gamma p\frac{\partial v}{\partial r} + \frac{\gamma\nu p v}{r} = 0$$

Here $\nu = 0$ for the plane case, $\nu = 1$ for the cylindrical one, and $\nu = 2$ for spherical symmetry.

2. We investigate the case when the spherical or cylindrical detonation front converges from infinity according to the rule